

# ON ASYMORPHISMS OF GROUPS

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**ABSTRACT.** Let  $G, H$  be groups and  $\kappa$  be a cardinal. A bijection  $f : G \rightarrow H$  is called an *asymorphism* if, for any  $X \in [G]^{<\kappa}$ ,  $Y \in [H]^{<\kappa}$ , there exist  $X' \in [G]^{<\kappa}$ ,  $Y' \in [H]^{<\kappa}$  such that for all  $x \in G$  and  $y \in H$ , we have  $f(Xx) \subseteq Y'f(x)$ ,  $f^{-1}(Yy) \subseteq X'f^{-1}(y)$ . For a set  $S$ ,  $[S]^{<\kappa}$  denotes the set  $\{S' \subseteq S : |S'| < \kappa\}$ .

Let  $\kappa$  and  $\gamma$  be cardinals such that  $\aleph_0 < \kappa \leq \gamma$ . We prove that any two Abelian groups of cardinality  $\gamma$  are  $\kappa$ -asymorphic, but the free group of rank  $\gamma$  is not  $\kappa$ -asymorphic to an Abelian group provided that either  $\kappa < \gamma$  or  $\kappa = \gamma$  and  $\kappa$  is a singular cardinal. It is known [7] that if  $\gamma = \kappa$  and  $\kappa$  is regular then any two groups of cardinality  $\kappa$  are  $\kappa$ -asymorphic.

## 1. INTRODUCTION

Following [5], [8], we say that a *ball structure* is a triple  $\mathcal{B} = (X, P, B)$ , where  $X, P$  are non-empty sets, and for all  $x \in X$  and  $\alpha \in P$ ,  $B(x, \alpha)$  is a subset of  $X$  which is called a *ball of radius  $\alpha$*  around  $x$ . It is supposed that  $x \in B(x, \alpha)$  for all  $x \in X$ ,  $\alpha \in P$ . The set  $X$  is called the *support* of  $\mathcal{B}$ ,  $P$  is called the *set of radii*.

Given any  $x \in X$ ,  $A \subseteq X$ ,  $\alpha \in P$ , we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha), \quad B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha).$$

A ball structure  $\mathcal{B} = (X, P, B)$  is called a *balleian* if

- for any  $\alpha, \beta \in P$ , there exist  $\alpha', \beta'$  such that, for every  $x \in X$ ,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- for any  $\alpha, \beta \in P$ , there exists  $\gamma \in P$  such that, for every  $x \in X$ ,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

- for any  $x, y \in X$ , there exists  $\alpha \in P$  such that  $y \in B(x, \alpha)$ .

We note that a balleian can be considered as an asymptotic counterpart of a uniform space, and could be defined [9] in terms of the entourages of the diagonal  $\Delta_X$  in  $X \times X$ . In this case a balleian is called a *coarse structure*. For categorical look at the balleians and coarse structures as "two faces of the same coin" see [2].

Let  $\mathcal{B} = (X, P, B)$ ,  $\mathcal{B}' = (X', P', B')$  be balleians. A mapping  $f : X \rightarrow X'$  is called a  $\prec$ -*mapping* if, for every  $\alpha \in P$ , there exists  $\alpha' \in P'$  such that, for every  $x \in X$ ,  $f(B(x, \alpha)) \subseteq B'(f(x), \alpha')$ .

A bijection  $f : X \rightarrow X'$  is called an *asymorphism* between  $\mathcal{B}$  and  $\mathcal{B}'$  if  $f$  and  $f^{-1}$  are  $\prec$ -mappings. In this case  $\mathcal{B}$  and  $\mathcal{B}'$  are called *asymorphic*.

Let  $\mathcal{B} = (X, P, B)$  be a balleian. Each subset  $Y$  of  $X$  defines a *subballeian*  $\mathcal{B}_Y = (Y, P, B_Y)$ , where  $B_Y(y, \alpha) = Y \cap B(y, \alpha)$ . A subset  $Y$  of  $X$  is called *large* if  $X = B(Y, \alpha)$ , for  $\alpha \in P$ . Two balleians  $\mathcal{B}$  and  $\mathcal{B}'$  with supports  $X$  and  $X'$  are called *coarsely equivalent* if there exist large subsets  $Y \subseteq X$  and  $Y' \subseteq X'$  such that the subballeians  $\mathcal{B}_Y$  and  $\mathcal{B}'_{Y'}$  are asymorphic. In the proof of Theorem 2, we use the following equivalent definition:  $\mathcal{B}$  and  $\mathcal{B}'$  are coarsely equivalent if there is a  $\prec$ -mapping  $f : X_1 \rightarrow X_2$  such that  $f(X_1)$  is large and, for every  $\alpha' \in P'$  there exists  $\alpha \in P$  such that  $f^{-1}(B'(f(x), \alpha')) \subseteq B(x, \alpha)$  for each  $x \in X$ .

We recall [6] that an ideal  $\mathcal{I}$  in the Boolean algebra of all subsets of a group  $G$  is a *group ideal* if  $AB^{-1} \in \mathcal{I}$  for any  $A, B \in \mathcal{I}$ . We suppose that  $\mathcal{I}$  contains all finite subsets of  $G$  and denote by  $(G, \mathcal{I})$  the ballean  $\mathcal{B} = (G, \mathcal{I}, B)$ , where  $P(g, A) = (A \cup \{e\})g$ ,  $e$  is the identity of  $G$ .

For an infinite group  $G$  and an infinite cardinal  $\kappa$ ,  $\kappa \leq |G|$ , we denote by  $[G]^{<\kappa}$  the group ideal  $\{A \subset G : |A| < \kappa\}$ .

We say that two groups  $G$  and  $H$  are  $\kappa$ -*asymorphic* ( $\kappa$ -*coarsely equivalent*) if the ballenans  $(G, [G]^{<\kappa})$  and  $(H, [H]^{<\kappa})$  are asymorphic (coarsely equivalent).

In the case  $\kappa = \aleph_0$ , we say that  $G$  and  $H$  are *finitary asymorphic* and *finitary coarsely equivalent* respectively. We note that finitely generated groups are finitary coarsely equivalent if and only if  $G$  and  $H$  are quasi-isometric [3, Chapter 4].

A classification of countable locally finite groups (each finite subset generates finite subgroup) up to finitary asymorphisms is obtained in [4]: there are continuum distinct types (see also [5, p. 103]) and each such group is finitary asymorphic to some direct product of finite cyclic groups. For coarse classifications of countable Abelian groups see [1]. Any two countable torsion Abelian groups are finitary coarsely equivalent.

This note is motivated by the following result [7, Theorem 3]. Let  $G, H$  be groups of cardinality  $\gamma$ ,  $\gamma > \aleph_0$  and let  $\kappa$  be a cardinal such that  $\aleph_0 < \kappa \leq \gamma$ . If  $\kappa = \gamma$  and  $\kappa$  is regular then  $G$  and  $H$  are  $\kappa$ -asymorphic.

What happens if  $\kappa < \gamma$  or  $\kappa = \gamma$  but  $\kappa$  is singular?

## 2. THEOREMS

**Theorem 1.** *Let  $G$  be an Abelian group of cardinality  $\gamma$ ,  $\gamma > \aleph_0$  and let  $\kappa$  be a cardinal such that  $\aleph_0 < \kappa \leq \gamma$ . Then  $G$  is  $\kappa$ -asymorphic to the free Abelian group  $A_\gamma$  of rank  $\gamma$ .*

**Theorem 2.** *Let  $G$  be an Abelian group of cardinality  $\gamma$ ,  $\gamma > \aleph_0$ . Then  $G$  is not  $\kappa$ -coarsely equivalent to the free group  $F_\gamma$  of rank  $\gamma$  provided that either  $\aleph_0 < \kappa < \gamma$  or  $\kappa = \gamma$  and  $\gamma$  is a singular cardinal. In particular,  $G$  and  $F_\gamma$  are not  $\kappa$ -asymorphic.*

## 3. PROOFS

*Proof of Theorem 1.* We choose a system of subgroups  $\{G_\alpha : \alpha < \gamma\}$  of  $G$  such that

- (1)  $G_0 = \{e\}$ ,  $G = \bigcup_{\alpha < \gamma} G_\alpha$ ;
- (2)  $G_\alpha \subset G_\beta$  for all  $\alpha < \beta < \gamma$ ;
- (3)  $G_\beta = \bigcup_{\alpha < \beta} G_\alpha$  for every limit ordinal  $\beta < \gamma$ ;
- (4)  $|G_{\alpha+1} \setminus G_\alpha| = \aleph_0$  for every  $\alpha < \gamma$ .

For each  $\alpha < \gamma$ , we fix some system  $X_\alpha$  of representatives of cosets of  $G_{\alpha+1} \setminus G_\alpha$  by  $G_\alpha$  so  $G_{\alpha+1} \setminus G_\alpha = X_\alpha G_\alpha$ .

We take an arbitrary element  $g \in G \setminus \{e\}$  and choose the smallest subgroup  $G_\gamma$  such that  $g \in G_\gamma$ . By (3),  $\alpha = \alpha_0 + 1$  for some  $\alpha_0 < \gamma$ . Then  $g = g_0 x_0$ ,  $g_0 \in G_{\alpha_0}$ ,  $x_0 \in X_{\alpha_0}$ . If  $g_0 \neq e$ , we repeat the argument for  $g_0$ : choose  $\alpha_1$  such that  $g_0 \in G_{\alpha_1+1} \setminus G_{\alpha_1}$  and write  $g_0 = g_1 x_1$ , where  $g_1 \in G_{\alpha_1}$ ,  $x_1 \in X_{\alpha_1}$  and so on. Since the set of ordinals less than  $\kappa$  is well ordered, after finite number of steps, we get

$$g = x_{s(g)} \dots x_1 x_0, \quad x_i \in X_{\alpha_i}, \quad i \in \{0, \dots, s(g)\}, \quad \alpha_0 > \alpha_1 > \dots > \alpha_{s(g)},$$

We observe that such a representation is unique and denote

$$\text{supt}(g) = \{\alpha_{s(g)}, \dots, \alpha_1, \alpha_0\}, \quad \text{supt}(e) = \emptyset$$

We identify  $A_\gamma$  with the direct product  $\times_{\alpha < \gamma} Z_\alpha$  of infinite cyclic groups and, for each  $\alpha < \gamma$ , fix some bijection  $f_\alpha : X_\alpha \rightarrow Z_\alpha \setminus \{e_\alpha\}$ ,  $e_\alpha$  is the identity of  $Z_\alpha$ . We define a bijection  $f : G \rightarrow A_\gamma$  putting  $f(e) = (e_\alpha)_{\alpha < \gamma}$  and, for  $g \in G \setminus \{e\}$ ,

$$f(g) = f(x_{s(g)} \dots x_1 x_0) = (f_{\alpha_{s(g)}}(x_{s(g)}), \dots, f_{\alpha_1}(x_1), f_{\alpha_0}(x_0)).$$

We show that  $f$  is an asymorphism between the ballenans  $(G, [G]^{<\kappa})$  and  $(A_\gamma, [A_\gamma]^{<\kappa})$ .

To show that  $f^{-1}$  is a  $\prec$ -mapping, we take  $a \in A_\gamma$ ,  $K \in [A_\gamma]^{<\kappa}$  and choose a set  $I \in [\gamma]^{<\kappa}$  such that  $K \subseteq \times_{\alpha \in I} Z_\alpha$ . We denote  $X = \times_{\alpha \in I} Z_\alpha$ ,  $b = pr_I a$ ,  $c = pr_{\gamma \setminus I} a$ . Then we have

$$\begin{aligned} f^{-1}(Ka) \setminus f^{-1}(Xc) &= f^{-1}(X)f^{-1}(c) = f^{-1}(X)(f^{-1}(b))^{-1}f^{-1}(b)f^{-1}(c) = \\ &= f^{-1}(X)(f^{-1}(b))^{-1}f^{-1}(a) \subseteq f^{-1}(X)(f^{-1}(X))^{-1}f^{-1}(a) \end{aligned}$$

and it suffices to note that  $f^{-1}(X)(f^{-1}(X))^{-1} \in [G]^{<\kappa}$ .

The verification that  $f$  is a  $\prec$ -mapping is more delicate. We take an arbitrary  $F \in [G]^{<\kappa}$  and denote by  $Y$  the smallest subgroup of  $G$  containing  $F$  and such that if  $g \in Y$  and  $\alpha \in \text{supt}(g)$  then  $X_\alpha \subseteq Y$ . We show that  $Y \in [G]^{<\kappa}$ . Indeed,  $Y$  can be obtained in the following way. For a subset  $A$  of  $G$ , we denote by  $\langle A \rangle$  the subgroup generated by  $A$  and  $h(A) = A \cup \bigcup \{X_\alpha : \alpha \in \text{supt}(g), g \in A\}$ . We put  $S_0 = \langle F \rangle$ ,  $Y_0 = h(S_0)$  and inductively  $S_{n+1} = \langle Y_n \rangle$ ,  $Y_{n+1} = h(S_{n+1})$ . Then  $Y = \bigcup_{n \in \omega} Y_n$ .

To conclude the proof, we take an arbitrary  $g \in G$ , put  $I = \bigcup_{g \in Y} \text{supt}(g)$ , and write  $g = g_0 g_1$  where  $\text{supt}(g_0) \subseteq I$ ,  $\text{supt}(g_1) \subseteq \gamma \setminus I$ . Then we have

$$f(Kg) \subseteq f(Yg_0g_1) \subseteq f(Yg_1) \subseteq f(Y)f(g_1) \subseteq f(Y)f(g).$$

*Proof of Theorem 2.* We are going to get a contradiction assuming only that there is a  $\prec$ -mapping  $f : G \rightarrow F_\gamma$  such that  $F_\gamma = Kf(G)$  for some  $K \in [F_\gamma]^{<\kappa}$ . In view of Theorem 1, we may suppose that  $G$  is a group of exponent 2. Since  $f$  is a  $\prec$ -mapping, for every  $a \in G$ , there exists  $K_a \in [F_\kappa]^{<\kappa}$  such that, for each  $x \in G$ , we have

$$f(a+x) \in K_a f(x).$$

We note that the family  $\{K_a : a \in G\}$  can be chosen so that, for some cardinal  $\delta$ ,  $\kappa \leq \delta < \gamma$ ,

$$(5) \quad |K_a| < \delta \text{ for each } a \in G.$$

If  $\kappa < \gamma$  then (5) is evident with  $\gamma = \kappa$ . We consider the case  $\gamma = \kappa$  and  $\kappa$  is singular. If (5) could not be satisfied then, for every  $\delta < \gamma$ , there exists  $a_\delta \in G$  such that

$$|\{f(a_\delta + x)f^{-1}(x) : x \in G\}| > \delta.$$

We denote  $Y_\delta = \{f(a_\delta + x)f^{-1}(x) : x \in G\}$  and use singularity of  $\gamma$  to choose a subset  $\Delta$  of  $\gamma$  such that  $|\Delta| < \gamma$  and the set  $\{|Y_\delta| : \delta \in \Delta\}$  is cofinal in  $\gamma$ . We put  $A = \{a_\delta : \delta \in \Delta\}$ . Since  $f$  is a  $\prec$ -mapping, there is a subset  $Y$  of  $F_\gamma$  such that  $|Y| = \kappa$  and  $\{f(A+x)f^{-1}(x) : x \in G\} \subseteq Y$ . Then we have got a contradiction with  $Y_\delta \subseteq Y$  for each  $\delta \in \Delta$ .

We consider  $F_\gamma$  as the group of all reduced group words over the alphabet  $\kappa$ . For  $g \in F_\gamma$ , we denote by  $\text{alph}f(g)$  the set of all letters  $\alpha < \gamma$  such that  $\alpha$  or  $\alpha^{-1}$  occurs in  $g$ , and for subset  $S$  of  $F_\gamma$ , we put  $\text{alph}(S) = \bigcup_{g \in S} \text{alph}f(g)$ .

Now we show how to find  $a, b \in G$  such that

- (6)  $\text{alph}f(a) \cap \text{alph}f(b) = \emptyset$ ;
- (7)  $\text{alph}f(a) \setminus \text{alph}K_b \neq \emptyset$
- (8)  $\text{alph}f(b) \cap \text{alph}K_a \neq \emptyset$

Since  $F_\gamma = Kf(G)$  for some  $K \in [F_\gamma]^{<\kappa}$ , we can choose a subset  $A \subseteq G$  such that  $|A| = \delta$  and  $|\text{alph}f(A)| = \delta$ . We take

$$\beta \in \gamma \setminus (\text{alph}f(A) \cup \bigcup_{a \in A} \text{alph}K_a)$$

and find  $b \in G$  such that  $f(b) = \beta$ . Since  $|K_\beta| < \delta$  and  $|\text{alph}f(A)| = \delta$ , there is  $a \in A$  such that  $\text{alph}f(a) \setminus \text{alph}K_b \neq \emptyset$ .

We put  $f(a+b) = z$  and note that

- (9)  $f(b) = f(a + (a+b)) \in K_a z$ ;
- (10)  $f(a) = f(b + (a+b)) \in K_b z$ .

We take  $u \in \text{alph}f(a) \setminus K_b$ ,  $v \in \text{alph}f(a) \setminus \text{alph}K_b$ . Then  $u \in \text{alph}(z)$ ,  $v \in \text{alph}(z)$ . If  $u$  occurs in  $z$  before  $v$  then, by (6), (10) is false. If  $v$  occurs in  $z$  before  $u$  then, by (6), (9) is false. These contradictions conclude the proof.

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